ABSTRACT CONVEXITY AND CONE-VEXING ABSTRACTIONS

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This talk is devoted to some origins of abstract convexity and a few vexing limitations on the range of abstraction in convexity. Convexity is a relatively recent subject. Although the noble objects of Euclidean geometry are mostly convex, the abstract notion of a convex set appears only after the Cantor paradise was founded. The idea of convexity feeds generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability.

1. Generation. Let $\overline{E}$ be a complete lattice E with the adjoint top $\top := +\infty$ and bottom $\perp := -\infty$. Unless otherwise stated, Y is usually a *Kantorovich space* which is a Dedekind complete vector lattice in another terminology. Assume further that H is some subset of E which is by implication a (convex) cone in E, and so the bottom of E lies beyond H. A subset U of H is *convex relative to* H or H-*convex*, in symbols $U \in \mathfrak{V}(H, \overline{E})$, provided that U is the H-*support set* $U_p^H := \{h \in H : h \leq p\}$ of some element p of $\overline{E}$.

Alongside the H-convex sets we consider the so-called H-convex elements. An element $p \in \overline{E}$ is H-*convex* provided that $p = \sup U_p^H$; i.e., p represents the supremum of the H-support set of p. The H-convex elements comprise the cone which is denoted by $\mathscr{C}(H, \overline{E})$. We may omit the references to H when H is clear from the context. It is worth noting that convex elements and sets are "glued together" by the *Minkowski diality* $\varphi : p \mapsto U_p^H$. This duality enables us to study convex elements and sets simultaneously.

Since the classical results by Fenchel [1] and Hörmander [2, 3] it has been well known that the most convenient and conventional classes of convex functions and sets are $\mathscr{C}(A(X), \overline{\mathbb{R}^X})$ and $\mathfrak{V}(X', \overline{\mathbb{R}^X})$. Here X is a locally convex space, X' is the dual of X, and $A(X)$ is the space of affine functions on X (isomorphic with $X' \times \mathbb{R}$).

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In the first case the Minkowski duality is the mapping $f \mapsto \operatorname{epi}(f^*)$ where

$$f^*(y) := \sup_{x\in X}(\langle y, x\rangle - f(x))$$

is the *Young–Fenchel transform* of f or the *conjugate function* of f . In the second case we prefer to write down the inverse of the Minkowski duality which sends U in $\mathfrak{V}(X', \overline{\mathbb{R}}^X)$ to the standard *support function*

$$\varphi^{-1}(U) : x \mapsto \sup_{y\in U}\langle y, x\rangle.$$

As usual, $\langle\cdot,\cdot\rangle$ stands for the canonical pairing of X' and X.

This idea of abstract convexity lies behind many current objects of analysis and geometry. Among them we list the "economical" sets with boundary points meeting the Pareto criterion, capacities, monotone seminorms, various classes of functions convex in some generalized sense, for instance, the Bauer convexity in Choquet theory, etc. It is curious that there are ordered vector spaces consisting of the convex elements with respect to narrow cones with finite generators. Abstract convexity is traced and reflected, for instance, in [4]–[9].

2. Separation. Consider cones K_1 and K_2 in a topological vector space X and put $\varkappa := (K_1, K_2)$. Given a pair $\varkappa$ define the correspondence $\Phi_\varkappa$ from X^2 into X by the formula

$$\Phi_\varkappa := \{(k_1, k_2, x) \in X^3 : x = k_1 - k_2 \in K_\imath \ (\imath := 1, 2)\}.$$

Clearly, $\Phi_\varkappa$ is a cone or, in other words, a conic correspondence.

The pair $\varkappa$ is *nonoblate* whenever $\Phi_\varkappa$ is open at the zero. Since $\Phi_\varkappa(V) = V \cap K_1 - V \cap K_2$ for every $V \subset X$, the nonoblateness of $\varkappa$ means that

$$\varkappa V := (V \cap K_1 - V \cap K_2) \cap (V \cap K_2 - V \cap K_1)$$

is a zero neighborhood for every zero neighborhood $V \subset X$. Since $\varkappa V \subset V - V$, the nonoblateness of $\varkappa$ is equivalent to the fact that the system of sets $\{\varkappa V\}$ serves as a filterbase of zero neighborhoods while V ranges over some base of the same filter.

Let $\Delta_n : x \mapsto (x, \dots, x)$ be the embedding of X into the diagonal $\Delta_n(X)$ of X^n. A pair of cones $\varkappa := (K_1, K_2)$ is nonoblate if and only if $\lambda := (K_1 \times K_2, \Delta_2(X))$ is nonoblate in X^2.

Cones K_1 and K_2 constitute a nonoblate pair if and only if the conic correspondence $\Phi \subset X \times X^2$ defined as

$$\Phi := \{(h, x_1, x_2) \in X \times X^2 : x_\imath + h \in K_\imath \ (\imath := 1, 2)\}$$

is open at the zero. Recall that a convex correspondence Φ from X into Y is open at the zero if and only if the Hörmander transform of

$X \times \Phi$ and the cone $\Delta_2(X) \times \{0\} \times \mathbb{R}^+$ constitute a nonoblate pair in $X^2 \times Y \times \mathbb{R}$.

Cones K_1 and K_2 in a topological vector space X are *in general position* provided that

(1) the algebraic span of K_1 and K_2 is some subspace $X_0 \subset X$; i.e., $X_0 = K_1 - K_2 = K_2 - K_1$;

(2) the subspace X_0 is complemented; i.e., there exists a continuous projection $P : X \to X$ such that $P(X) = X_0$;

(3) K_1 and K_2 constitute a nonoblate pair in X_0.

Let σ_n stand for the rearrangement of coordinates

$$\sigma_n : ((x_1, y_1), \dots, (x_n, y_n)) \mapsto ((x_1, \dots, x_n), (y_1, \dots, y_n))$$

which establishes an isomorphism between $(X \times Y)^n$ and $X^n \times Y^n$.

Sublinear operators $P_1, \dots, P_n : X \to E \cup \{+\infty\}$ are *in general position* if so are the cones $\Delta_n(X) \times E^n$ and $\sigma_n(\mathrm{epi}(P_1) \times \dots \times \mathrm{epi}(P_n))$. A similar terminology applies to convex operators.

Given a cone $K \subset X$, put

$$\pi_E(K) := \{T \in \mathscr{L}(X, E) : Tk \le 0 \ (k \in K)\}.$$

We readily see that $\pi_E(K)$ is a cone in $\mathscr{L}(X, E)$.

Theorem. *Let $K_1, \dots, K_n$ be cones in a topological vector space X and let E be a topological Kantorovich space. If $K_1, \dots, K_n$ are in general position then*

$$\pi_E(K_1 \cap \dots \cap K_n) = \pi_E(K_1) + \dots + \pi_E(K_n).$$

This formula opens a way to various separation results.

Sandwich Theorem. *Let $P, Q : X \to E \cup \{+\infty\}$ be sublinear operators in general position. If $P(x) + Q(x) \ge 0$ for all $x \in X$ then there exists a continuous linear operator $T : X \to E$ such that*

$$-Q(x) \le Tx \le P(x) \quad (x \in X).$$

Many efforts were made to abstract these results to a more general algebraic setting and, primarily, to semigroups. The relevant separation results are collected in [10].

3. Calculus. Consider a Kantorovich space E and an arbitrary nonempty set $\mathfrak{A}$. Denote by $l_\infty(\mathfrak{A}, E)$ the set of all order bounded mappings from $\mathfrak{A}$ into E; i.e., $f \in l_\infty(\mathfrak{A}, E)$ if and only if $f : \mathfrak{A} \to E$ and the set $\{f(\alpha) : \alpha \in \mathfrak{A}\}$ is order bounded in E. It is easy to verify that $l_\infty(\mathfrak{A}, E)$ becomes a Kantorovich space if endowed with the coordinatewise algebraic operations and order. The operator $\varepsilon_{\mathfrak{A},E}$ acting from $l_\infty(\mathfrak{A}, E)$ into E by the rule

$$\varepsilon_{\mathfrak{A},E} : f \mapsto \sup\{f(\alpha) : \alpha \in \mathfrak{A}\} \quad (f \in l_\infty(\mathfrak{A}, E))$$

is called the *canonical sublinear operator* given $\mathfrak{A}$ and E. We often write $\varepsilon_{\mathfrak{A}}$ instead of $\varepsilon_{\mathfrak{A},E}$ when it is clear from the context what Kantorovich space is meant. The notation ε_n is used when the cardinality of $\mathfrak{A}$ equals n and we call the operator ε_n *finitely-generated.*

Let X and E be ordered vector spaces. An operator $p : X \to E$ is called *increasing* or *isotonic* if for all $x_1, x_2 \in X$ from $x_1 \leq x_2$ it follows that $p(x_1) \leq p(x_2)$. An increasing linear operator is also called *positive.* As usual, the collection of all positive linear operators in the space $L(X, E)$ of all linear operators is denoted by $L^+(X, E)$. Obviously, the positivity of a linear operator T amounts to the inclusion $T(X^+) \subset E^+$, where $X^+ := \{x \in X : x \geq 0\}$ and $E^+ := \{e \in E : e \geq 0\}$ are the *positive cones* in X and E respectively. Observe that every canonical operator is increasing and sublinear, while every finitely-generated canonical operator is order continuous.

Recall that $\partial p := \partial p(0) = \{T \in L(X, E) : (\forall x \in X)\ Tx \leq p(x)\}$ is the *subdifferential* at the zero or *support set* of a sublinear operator p.

Consider a set $\mathfrak{A}$ of linear operators acting from a vector space X into a Kantorovich space E. The set $\mathfrak{A}$ is *weakly order bounded* if the set $\{\alpha x : \alpha \in \mathfrak{A}\}$ is order bounded for every $x \in X$. We denote by $\langle\mathfrak{A}\rangle x$ the mapping that assigns the element $\alpha x \in E$ to each $\alpha \in \mathfrak{A}$, i.e. $\langle\mathfrak{A}\rangle x : \alpha \mapsto \alpha x$. If $\mathfrak{A}$ is weakly order bounded then $\langle\mathfrak{A}\rangle x \in l_\infty(\mathfrak{A}, E)$ for every fixed $x \in X$. Consequently, we obtain the linear operator $\langle\mathfrak{A}\rangle : X \to l_\infty(\mathfrak{A}, E)$ that acts as $\langle\mathfrak{A}\rangle : x \mapsto \langle\mathfrak{A}\rangle x$. Associate with $\mathfrak{A}$ one more operator

$$p_{\mathfrak{A}} : x \mapsto \sup\{\alpha x : \alpha \in \mathfrak{A}\} \quad (x \in X).$$

The operator $p_{\mathfrak{A}}$ is sublinear. The support set $\partial p_{\mathfrak{A}}$ is denoted by $\operatorname{cop}(\mathfrak{A})$ and referred to as the *support hull* of $\mathfrak{A}$. These definitions entail the following

THEOREM. *If p is a sublinear operator with $\partial p = \operatorname{cop}(\mathfrak{A})$ then $P = \varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle$. Assume further that $p_1 : X \to E$ is a sublinear operator and $p_2 : E \to F$ is an increasing sublinear operator. Then*

$$\partial(p_2 \circ p_1) = \left\{T \circ \langle\partial p_1\rangle : T \in L^+(l_\infty(\partial p_1, E), F) \ \wedge\ T \circ \Delta_{\partial p_1} \in \partial p_2\right\}.$$

Furthermore, if $\partial p_1 = \operatorname{cop}(\mathfrak{A}_1)$ and $\partial p_2 = \operatorname{cop}(\mathfrak{A}_2)$ then

$$\begin{aligned}\partial(p_2 \circ p_1) = \big\{T \circ \langle\mathfrak{A}_1\rangle : T \in L^+(l_\infty(\mathfrak{A}_1, E), F)\\ \wedge\ \left(\exists \alpha \in \partial\varepsilon_{\mathfrak{A}_2}\right)\ T \circ \Delta_{\mathfrak{A}_1} = \alpha \circ \langle\mathfrak{A}_2\rangle\big\}.\end{aligned}$$

More details on subdifferential calculus and applications to optimality are collected in [11].

4. Approximation. Study of stability in abstract convexity is accomplished sometimes by introducing various epsilons in appropriate

places. One of the earliest attempts in this direction is connected with the classical Hyers–Ulam stability theorem for ε-convex functions. The most recent results are collected in [12]. Exact calculations with epsilons and sharp estimates are sometimes bulky and slightly mysterious. Some alternatives are suggested by actual infinities, which is illustrated with the conception of *infinitesimal optimality*.

Assume given a convex operator $f : X \to E \cup +\infty$ and a point $\overline{x}$ in the effective domain $\operatorname{dom}(f) := \{x \in X : f(x) < +\infty\}$ of f. Given $\varepsilon \geq 0$ in the positive cone E_+ of E, by the *ε-subdifferential* of f at $\overline{x}$ we mean the set

$$\partial^{\varepsilon} f(\overline{x}) := \big\{T \in L(X,E) : (\forall x \in X)(Tx - Fx \leq T\overline{x} - f\overline{x} + \varepsilon)\big\},$$

with $L(X,E)$ standing as usual for the space of linear operators from X to E.

Distinguish some downward-filtered subset $\mathscr{E}$ of E that is composed of positive elements. Assuming E and $\mathscr{E}$ standard, define the *monad* $\mu(\mathscr{E})$ of $\mathscr{E}$ as $\mu(\mathscr{E}) := \bigcap\{[0,\varepsilon] : \varepsilon \in {}^{\circ}\mathscr{E}\}$. The members of $\mu(\mathscr{E})$ are *positive infinitesimals* with respect to $\mathscr{E}$. As usual, ${}^{\circ}\mathscr{E}$ denotes the external set of all standard members of E, the *standard part* of $\mathscr{E}$.

We will agree that the monad $\mu(\mathscr{E})$ is an external cone over ${}^{\circ}\mathbb{R}$ and, moreover, $\mu(\mathscr{E}) \cap {}^{\circ}E = 0$. In application, $\mathscr{E}$ is usually the filter of order-units of E. The relation of *infinite proximity* or *infinite closeness* between the members of E is introduced as follows:

$$e_1 \approx e_2 \leftrightarrow e_1 - e_2 \in \mu(\mathscr{E}) \wedge e_2 - e_1 \in \mu(\mathscr{E}).$$

Since

$$\bigcap_{\varepsilon \in {}^{\circ}\mathscr{E}} \partial_{\varepsilon} f(\overline{x}) = \bigcup_{\varepsilon \in \mu(\mathscr{E})} \partial_{\varepsilon} f(\overline{x});$$

therefore, the external set on both sides is the so-called *infinitesimal subdifferential* of f at $\overline{x}$. We denote this set by $Df(\overline{x})$. The elements of $Df(\overline{x})$ are *infinitesimal subgradients* of f at $\overline{x}$. If the zero oiperator is an infinitesimal subgradient of f at $\overline{x}$ then $\overline{x}$ is called an *infinitesimal minimum point* of f. We abstain from indicating $\mathscr{E}$ explicitly since this leads to no confusion.

Theorem. *Let $f_1 : X \times Y \to E \cup +\infty$ and $f_2 : Y \times Z \to E \cup +\infty$ be convex operators. Suppose that the convolution $f_2 \vartriangle f_1$ is infinitesimally exact at some point (x,y,z); i.e., $(f_2 \vartriangle f_1)(x,y) \approx f_1(x,y) + f_2(y,z)$. If, moreover, the convex sets* $\operatorname{epi}(f_1, Z)$ *and* $\operatorname{epi}(X, f_2)$ *are in general position then*

$$D(f_2 \vartriangle f_1)(x,y) = Df_2(y,z) \circ Df_1(x,y).$$

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